Question 1. Let f be analytic inside domain C. Prove that the number of zeros of f inside C equals $\frac{1}{2\pi}\Delta_C \arg(f)$ where $\Delta_C \arg(f)$ is the total change in argument of $f(z)$ when moving along C.

Let $N(T)$ be the number of zeros of $\zeta(z)$ in the rectangle $0 < R(z) < 1$ and $0 < R(z) < T$.

Prove that

$$N(T) = \frac{T}{2\pi}\log\frac{T}{2\pi} - \frac{T}{2\pi} + O(\log T).$$

Hint: Let $f = \xi$, C be the rectangle with vertices 2,2 $+ iT, +iT,$ _1 and use above.

Question 2. Let x be a *primitive* character modulo q (primitive means $x^k \neq 1$ for any $k < \emptyset(q)$; all the results we discussed in class regarding L - functions hold only for primitive characters as non- primitive characters are characters also of some $p < q, p|q$).
Assume that $x(-1) = 1$. Let

$$\xi(z,x) = (q/\pi)^{z/2}\,\Gamma(z/2)\,L(z,x).$$

Prove that $L(z,x)$ has infinitely many zeros in the strip $0 < R(z) < T$ and

$$\xi(z,x) = e^{Az+B\,TT} \quad {}_{p}\left(1 - \frac{z}{p}\right).\, e^{z/p},$$

Where $p$ranges over non- trivial zeros of $L(z,x)$. You may assume that $\xi(z) = \xi(1-z)$.

Question 3. Define

$$G_k(z) = \sum_{(m,n)\neq(0,0)} \frac{1}{(mz+n)^k},$$

for $k \geq 3$. prove that

$$G_k(rz) = (ez+d)^k G_k(z),$$

for $T \in SL_2(Z). T = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

Question 4. Derive Laurent series expansion of $\zeta(z)$ around $z = 1$. Since $\zeta(z)$ is meromorphic over the complex plane with a simple pole at $z = 1$,this expansion will be valid over the entire complex plane.

Question 5. Prove that

$$\zeta(z) = \sum_{n=1}^{m=1} \frac{1}{n^2} + \frac{m^{1-z}}{z-1} - z \int_m^{\infty} \frac{t - |t| - 1/2}{t^{z+1}}\, dt\,,$$

for all z with $\Re(z) > 0$

Question 6. In this question, we study the behavior of $\zeta(z)$ inside the rectangle R_T defined by points $2 - iT, 2 + iT, \alpha_0 + iT, \alpha_0 - iT$ with $\alpha_0 = 1 - \frac{1}{\log T}$ and $T \geq e^2$.

- First show that

  $$\left|\zeta(z) - \frac{1}{z-1}\right| = 0(\log T)$$

  for any $z \in R_T$. use this to show that

  $$|\zeta(z)| = 0(\log T)$$

  for any $z \in R_T$.

- Next show that

  $$\left|\zeta'(z) - \frac{1}{(z-1)^2}\right| = 0(\log^2 T)$$

  for any $z \in R_T$. use this to show that

  $$|\zeta'(z)| = 0(\log^2 \mathrm{T})$$

  for any $z \in R_T$.